Assignment 1

Exercise 1.1

2. Which of the following operators are linear?

(a) $\mathscr{L}u = u_x + xu_y$

(b) $\mathscr{L}u = u_x + uu_y$

(c) $\mathscr{L}u = u_x + u_y^2$

(d) $\mathscr{L}u = u_x + u_y + 1$

(e) $\mathscr{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$

(b) $u_t - u_{xx} + xu = 0$

(c) $u_t - u_{xxt} + uu_x = 0$

(d) $u_{tt} - u_{xx} + x^2 = 0$

(e) $iu_t - u_{xx} + u/x = 0$

(f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$

(g) $u_x + e^y u_y = 0$

(h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.

11. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

12. Verify by direct substitution that

$$u_n(x, y) = \sin nx \sinh ny$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

Exercise 1.2

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

2. Solve the equation $3u_y + u_{xy} = 0$. (Hint: Let $v = u_y$.)

3. Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

5. Solve the equation $\sqrt{1-x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

6. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

(b) In which region of the xy plane is the solution uniquely determined?

7. Solve $au_x + bu_y + cu = 0$.

8. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.

9. Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f ds \ + \ g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (Hint: Use the coordinate method.)

11. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

Exercise 1.3

1. Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity.

5. Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

6. Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

7. Solve Exercise 6 in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2 + y^2 + z^2}$. Derive the equation $u_t = k(u_{rr} + 2u_r/r)$.

9. This is an exercise on the divergence theorem

$$\iiint_D \nabla \cdot \mathbf{F} d\mathbf{x} = \iint_{\text{bdy}D} \mathbf{F} \cdot \mathbf{n}$$

valid for any bounded domain D in space with boundary surface bdy D and unit outward normal vector $\mathbf{n}$. If you never learned it, see Section A.3. It is crucial that D be bounded. As an exercise, verify it in the following case by calculating both sides separately: $\mathbf{F} = r^2\mathbf{x}$, $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r^2 = x^2 + y^2 + z^2$, and $D =$the ball of radius a and center at the origin.

10. If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x}$, show that

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f} d\mathbf{x} = 0.$$

(Hint: Take D to be a large ball, apply the divergence theorem, and let its radius tend to infinity.)

Exercise 1.4

1. By trial and error, find a solution of the diffusion equation $u_t = u_{xx}$with the initial condition $u(x, 0) = x^2$.

2. (a) Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition $\partial u/\partial x = 0$. (Use Fouriers law.)

(b) Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$. (Use Ficks law.)

(c) Showthat the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition $\partial u/\partial n = 0$.

3. A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(\mathbf{x})$. Find the steady-state temperature that it reaches after a long time. (Hint: No heat is gained or lost.)

Exercise 1.5

1. Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$

$$u(0) = 0 \quad \text{and} \quad u(L) = 0$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on L?

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$

$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.

(a) Is the solution *unique*? Explain.

(b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

3. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D$$

(a) What can we surely add to any solution to get another solution? So we dont have uniqueness.

(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)dx\ dy\ dz = 0$$

is a necessary condition for the Neumann problem to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Exercise 1.6

1. What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

2. Find the regions in the xy plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

3. Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (*rotationally invariant*) have the form $a(u_{xx} + u_{yy}) + bu = 0$.

4. What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

5. Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x + \beta y}$ and then a change of scale $y' = \gamma y$.

6. Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (Hint: Substitute $v = u_y$.)

(c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?

Suggested Solution to Assignment 1

Exercise 1.1

2. By the definition of linearity for operators in 1.1(3), the operators in (a) and (e) are linear, others are not linear. □

3. (a) order 2 with u_{xx}, linear inhomogeneous; (b) order 2 with u_{xx}, linear homogeneous; (c) order 3 with u_{xxt}, nonlinear; (d) order 2 with with u_{tt}, u_{xx}, linear inhomogeneous; (e) order 2 with u_{xx}, linear homogeneous; (f) order 1 with u_x and u_y, nonlinear; (g) order 1 with u_x and u_y, linear homogeneous; (h) order 4 with u_{xxxx}, nonlinear. □

4. Suppose that $\mathscr{L}u_1 = g$ and $\mathscr{L}u_2 = g$ and let $u = u_1 - u_2$, then $\mathscr{L}u = \mathscr{L}u_1 - \mathscr{L}u_2 = g - g = 0$, where the operator L is linear. □

11. Let $u(x, y) = f(x)g(y)$, then by direct calculation we have

$$\begin{aligned} u(x,y)u_{xy}(x,y) &= f(x)g(y)f'(x)g'(y) \\ &= f'(x)g(y)f(x)g'(y) \\ &= u_x(x,y)u_y(x,y) \end{aligned}$$

Hence, $uu_{xy} = u_x u_y$ is verified. □

12. Let $u_n(x, y) = \sin nx \sinh ny$, then for $n > 0$,

$$u_{xx} + u_{yy} = -n^2 \sin(nx)\sinh(ny) + n^2 \sin(nx)\sinh(ny) = 0.$$

Thus, $u_{xx} + u_{yy} = 0$ is verified. □

Exercise 1.2

1. Using the characteristic curve method or the coordinate method, we have $u(t, x) = f(3t-2x)$. □. Setting $t = 0$ yields the equation $f(-2x) = \sin x$. Letting $w = -2x$ yields $f(w) = -\sin(w/2)$. Therefore, $u(t, x) = \sin(x - 3t/2)$. □

2. Let $v = u_y$, then $3v + v_x = 0$. Thus we have $v(x, y) = f(y)e^{-3x}$, i.e., $u_y(x, y) = f(y)e^{-3x}$, which implies $u(x, y) = F(y)e^{-3x} + g(x)$, where both F and g are arbitrary (differentiable) functions. □

3. The characteristic curves satisfy the ODE: $dy/dx = 1/(1 + x^2)$, which implies $y = \arctan x + C$. Thus $u(x, y) = f(y - \arctan x)$. We omit the easy figure here. □

5. The characteristic curves satisfy the ODE: $dy/dx = 1/\sqrt{1-x^2}$, which implies $y = \arcsin x + C$. Thus $u(x, y) = f(y - \arcsin x)$. Setting $x = 0$ yields the equation $f(y) = y$, and then $u(x, y) = y - \arcsin x$. □

6. (a)The characteristic curves satisfy the ODE: $dy/dx = x/y$, which implies $y^2 = x^2 + C$ and then $u(x, y) = f(y^2 - x^2)$. Setting $x = 0$ yields the equation $f(y^2) = e^{-y^2}$. Letting $w = y^2$ yields $f(w) = e^{-w}$ and $u(x, y) = e^{x^2-y^2}$. (b)Please see the following figure. □

7. Change variables to $x' = ax + by, y' = bx - ay$. By the chain rule,

$$u_x = au_{x'} + bu_{y'}, u_y = bu_{x'} - au_{y'}$$

. We have $(a^2 + b^2)u_{x'} + cu = 0$ which implies $u(x, y) = f(y)e^{-cx/(a^2+b^2)}$ and then $u(x, y) = f(bx - ay)e^{-c(ax+by)/(a^2+b^2)}$, where f is a arbitrary (differentiable) function. □

8. Note that $u(x,y) = e^{x+2y}/4$ is a special solution of te inhomogeneous equation, and by the result of 1.2.8 above, the general solution of the corresponding homogeneous equation is $f(x-y)e^{-(x+y)/2}$. Thus the general solution of $u_x + u_y + u = e^{x+2y}$ is

$$u(x,y) = f(x-y)e^{-(x+y)/2} + e^{x+2y}/4,$$

where f is a arbitrary function. Let $y = 0$, and then we have $f(x)e^{-x/2} + e^{x/4} = 0$, i.e. $f(x) = -e^{3x/2}/4$. So the solution is $u(x,y) = (e^{x+2y} - e^{x-2y})/4$. □

9. By changing variables,$x' = ax + by, y' = bx - ay$. The original equation is equivalent to the following form

$$(a^2+b^2)u_{x'} = f(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}).$$

Therefore, we have the general solution to the above equation is

$$u(x',y') = \frac{1}{a^2+b^2}\int_0^{x'} f(\frac{as'+by'}{a^2+b^2}, \frac{bs'-ay'}{a^2+b^2})\, ds' + g(y'),$$

where we let $g(y') = u(0,y')$, and g is a arbitrary function. Returning back to the original parameters, the integral changes to be the integral along the line

$$L = \{(m,n); 0 \le s' = am + bn \le ax + by, y' = bm - an = bx - ay\}.$$

When denoting s the parameter of arc length, we have $ds = \sqrt{(dm)^2 + (dn)^2}$. Note that along the line L the condition $bm - an = bx - ay$ is satisfied. Thus, $b(dm) - a(dn) = 0$, and then $ds = \frac{\sqrt{a^2+b^2}}{a}dm$, $ds' = a(dm) + b(dn) = \frac{a^2+b^2}{a}$. Hence, the solution is

$$u(x,y) = \frac{1}{(a^2+b^2)^{1/2}}\int_L f ds + g(bx - ay),$$

where L is shown above(actually, the line segment is not from the y axis). □

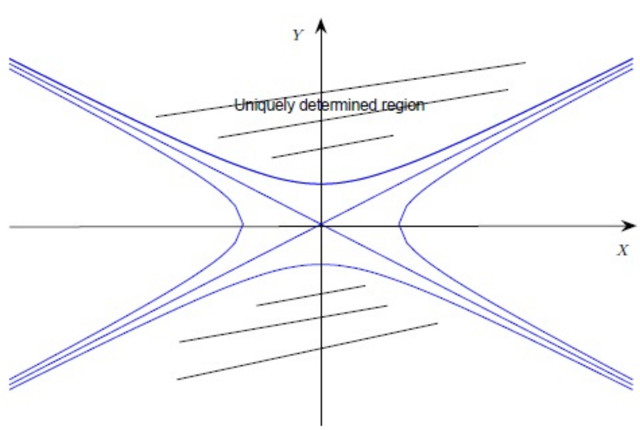
11. Using Coordinate Method, we change variables $x' = x + 2y, y' = 2x - y$, then the original equation is changed into

$$5u_{x'} + y'u = x'y'.$$

Note that $u(x',y') = x' - \frac{5}{y'}$ is a special solution and $u(x',y') = f(y')e^{-(x'y'/5)}$ is the general solution of the corresponding homogeneous equation. Hence the general solution of original equation is

$$u(x,y) = f(2x-y)e^{-\frac{(x+2y)(2x-y)}{5}} + x + 2y - \frac{5}{2x-y}$$

where f is an arbitrary (differentiable) function. □



Exercise 1.3

1. According to Example 2, we only need to add the resistence in the transverse equation. Under the assumption that the resistence is proportional to the velocity, the transverse equation becomes

$$\frac{Tu_x}{\sqrt{1+u_x^2}}\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} -ku_t dx = \int_{x_0}^{x_1} \rho u_{tt} dx$$

where $k > 0$ is a coefficient depending on the property of the medium (e.g. the density of the medium). Note that the direction of resistence should be opposite to the velocity, thus we have the negative sign before k. The equation, differentiated, says that

$$(Tu_x)_x - ku_t = \rho u_{tt}$$

That is,

$$u_{tt} - c^2 u_{xx} + ru_t = 0$$

where $c = \sqrt{\frac{T}{\rho}}, r = \frac{k}{\rho} > 0$.

5. Let $u(x,t)$ be the concentration (mass per unit length) of the dye at position x of the pipe at time t. The mass of dye is $M(t) = \int_{x_0}^{x} u(y,t)dy$, so $\frac{\partial M}{\partial t} = \int_{x_0}^{x} u_t(y,t)dy$. Then by the Fick's law,

$$\frac{\partial M}{\partial t} = \text{flow in} - \text{flow out} = V(u(x_0,t) - u(x,t)) + ku_x(x,t) - ku_x(x_0,t).$$

Differentiating with respect to x, we get $u_t = ku_{xx} - Vu_x$. □

6. Since the heat flow depends only on t and on the distance $r = \sqrt{x^2+y^2}$ to the axis of the cylinder,

$$u(x,y,z,t) = u(\sqrt{x^2+y^2}, t) = u(r,t).$$

Then by the chain rule,

$$\begin{aligned} &u_x = u_r x/r, && u_y = u_r y/r, && u_z = 0,\\ &u_{xx} = u_{rr}x^2/r^2 + u_r(r^2-x^2)/r^3, && u_{yy} = u_{rr}y^2/r^2 + u_r(r^2-y^2)/r^3, && u_{zz} = 0. \end{aligned}$$

Therefore, $u_t = k(u_{xx}+u_{yy}+u_{zz}) = k(u_{rr} + u_r/r)$. □

7. Since the heat flow depends only on t and on the distance $r = \sqrt{x^2+y^2+z^2}$ to the cylinder,

$$u(x,y,z,t) = u(\sqrt{x^2+y^2+z^2}, t) = u(r,t).$$

Then by the chain rule,

$$\begin{aligned} &u_x = u_r x/r, && u_y = u_r y/r, && u_z = u_r z/r,\\ &u_{xx} = \frac{u_{rr}x^2}{r^2} + \frac{u_r(r^2-x^2)}{r^3}, && u_{yy} = \frac{u_{rr}y^2}{r^2} + \frac{u_r(r^2-y^2)}{r^3}, && u_{zz} = \frac{u_{rr}z^2}{r^2} + \frac{u_r(r^2-z^2)}{r^3}. \end{aligned}$$

Therefore, $u_t = k(u_{xx}+u_{yy}+u_{zz}) = k(u_{rr} + 2u_r/r)$. □

9. Denote $\mathbf{F} = (F_1, F_2, F_3)$. Note that $\frac{\partial F_1}{\partial x} = r^2 + 2x^2$. Thus, we have

$$\iiint_D \nabla\cdot\mathbf{F}d\mathbf{x} = \int_0^a \iint_{\partial B(0,r)} \nabla\cdot\mathbf{F}\; dSdr = \int_0^a \iint_{\partial B(0,r)} 5r^2\; dSdr = 4\pi\int_0^a 5r^4 dr = 4\pi a^5,$$

$$\iint_{\text{bdy}D} \mathbf{F}\cdot\mathbf{n}dS = 4\pi a^2\; a^3 = 4\pi a^5,$$

where $\partial B(0,r)$ denotes the ball of radius r centered at O. □

10. By the divergence theorem, we have

$$\iiint_{|x|\le R} \nabla\cdot\mathbf{f}d\mathbf{x} = \iint_{|x|=R}\mathbf{f}\cdot\mathbf{n}dS \le \iint_{|x|=R}|\mathbf{f}|dS \le \iint_{|x|=R} 1/|\mathbf{x}|^3 dS = 4\pi/R.$$

Hence,

$$\iiint_{\text{all space}} \nabla\cdot\mathbf{f}d\mathbf{x} = \lim_{R\to\infty}\iiint_{|x|\le R}\nabla\cdot\mathbf{f}d\mathbf{x} \le \lim_{R\to\infty} 4\pi/R = 0. \quad \square$$

Exercise 1.4

1. Setting $u(x,t) = f(t)+x^2$ yields the equations $f(t) = 2$ and $f(0) = 0$. Hence, $f(t) = 2t$ and $u(x,t) = 2t+x^2$ is a solution of the diffusion equation. $\square$

2. (a) No heat flows across the boundary, by the Fouriers law, we have $\partial u/\partial x = 0$;
   (b) No gas flows across the boundary, by the Ficks law, we have $\partial u/\partial x = 0$;
   (c) No heat or gas flows across the boundary, by the Fouriers or Ficks law, we have $\partial u/\partial n = 0$. $\square$

3. After long time, if this homogeneous body reaches a steady state, then $\partial_t u = 0$, therefore, $u_{xx} = 0$. Since it is insulated, therefore, we have $u \equiv \mathit{constant}$. So the steady-state temperature is $\frac{\int_D f d\mathbf{x}}{\int_D d\mathbf{x}}$. $\square$

Exercise 1.5

1. The general solution of the ODE: $\frac{d^2u}{dx^2} + u = 0$ is $u(x) = C_1\cos x + C_2\sin x$. Hence, to satisfy the boundary conditions,

$$C_1 = 0 \quad\text{and}\quad C_1\cos(L) + C_2\sin(L) = 0.$$

Therefore, $C_1 = 0$ and $C_2\sin(L) = 0$. So the solution $u \equiv 0$ if and only if L is not an integer multiple of π. $\square$

2. (a) The solution is not unique. Indeed, if there exists a solution u_0, then $u_0 + C(e^{-x} - 2)$ is also a solution of equation for any constant C.
   (b) The solution does not necessarily exist, since the condition that $f(x)$ must satisfy for the existence is:

$$\int_0^l f(x)dx = \int_0^l [u''(x) + u'(x)]\,dx = [u'(l) + u(l)] - [u'(0) + u(0)] = 0. \qquad \square$$

3. The general solution of $u''(x) = 0$ is $u(x) = ax + b$, where a and b are constants. Hence, when we do the $+$ case, a and b have to satisfy $a + kb = 0$ and $a + k(a+b) = 0$, and then solution(s) of the boundary problem would be

$$u(x) = \begin{cases} 0 & \text{if } k \ne 0 \\ b & \text{if } k = 0 \end{cases};$$

when we do the $-$ case, the solution(s) of the boundary problem would be

$$u(x) = \begin{cases} 0 & \text{if } k \ne 0, 2 \\ b & \text{if } k = 0 \\ -2bx + b & \text{if } k = 2 \end{cases}.$$

If $k = 2$, the boundary problem is unique for the $+$ case, but not for the $-$ case. $\square$

4. (a) Adding a constant C to a solution will give another solution, so we do not have uniqueness if there is a solution;
   (b) Integrating $f(x,y,z)$ on D and using the divergence theorem, we obtain

$$\iiint_D f(x,y,z)dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla\cdot\nabla u dxdydz = \iint_{\partial D} \nabla u \cdot n dS = 0$$

   (c) For heat flow or diffusion, u is a physical quantity in terms of time t. The equation here can only describe the derivatives of u with respect to (x,y,z). So (a) shows that u up to a constant has the same derivatives with respect to (x,y,z).
   Since for heat flow and diffusion $u_t = k\Delta u = kf(x,y,z)$, (b) shows that to satisfy the boundary condition(insulated solid or impermeable container), the change of the whole heat energy or the whole substance with respect to time, which is proportion to $\iiint_D \frac{\partial u}{\partial t} dxdydz = k\iiint_D f(x,y,z)dxdydz$, has to be 0. □

Exercise 1.6

1. Indeed, we check the sign of the "discriminant" $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$.
   (a) $\mathscr{D} = [(-1-3)/2]^2 - 1\cdot 1 = 3 > 0$, so it is hyperbolic.
   (b) $\mathscr{D} = [6/2]^2 - 9\cdot 1 = 0$, so it is parabolic. □

2. Indeed, its discriminant is

$$\mathscr{D} = (xy)^2 - (1+x)(-y^2) = (x^2+x+1)y^2 = [(x+1/2)^2 + 3/4]y^2,$$

   So it is hyperbolic in $\{y \neq 0\}$, parabolic on $\{y = 0\}$, and elliptic nowhere. We omit the easy figure here. □

3. In the equations of the form (1), suppose $A = (a_{ij})$, $n = (a_i)$ and $b = a_0$. Denote $B = (b_{ij})$ as the matrix of the rotation. Therefore, the new variables (ξ,η) satisfy $(\xi,\eta)^T = B(x,y)^T$, and the new coefficients satisfy $A' = BAB^T$, $n' = nB^T$ and $b' = b$. So the rotationally invariant equations have to satisfy

$$A = BAB^T, \quad n' = nB^T \quad \forall \text{ normal matrix } B.$$

   Thus, A is a unit matrix multiple of a constant a, and $n = 0$. So all rotationally invariant equations of the form (1) have the form $a(u_{xx} + u_{yy}) + bu = 0$. □

4. It is parabolic since its discriminant $\mathscr{D} = (-4/2)^2 - 1\cdot 4 = 0$. By direct subtitution, if $u(x+y) = f(y+2x) + xg(y+2x)$, then $u_{xx} = 4f''(y+2x) + 4xg''(y+2x) + 4g'(y+2x)$, $u_{xy} = 2f''(y+2x) + 2xg''(y+2x) + g'(y+2x)$ and $u_{yy} = f''(y+2x) + xg''(y+2x)$, and then the equation is satisfied. □

5. Let $u = ve^{(\alpha x+\beta y)}$, then

$$u_x = (v_x + \alpha v)e^{(\alpha x+\beta y)} \qquad u_y = (v_y + \beta v)e^{(\alpha x+\beta y)}$$
$$u_{xx} = (v_{xx} + 2\alpha v_x + \alpha^2 v)e^{(\alpha x+\beta y)} \qquad u_{yy} = (v_{yy} + 2\beta v_y + \beta^2 v)e^{(\alpha x+\beta y)}$$

   Hence, by direct substituting,

$$(v_{xx} + 2\alpha v_x + \alpha^2 v) + 3(v_{yy} + 2\beta v_y + \beta^2 v) - 2(v_x + \alpha v) + 24(v_y + \beta v) + 5v = 0,$$

$$v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v = 0.$$

   Let $\alpha = 1$ and $\beta = -4$, the equation turns out to be $v_{xx} + 3v_{yy} - 44v = 0$. By setting $x' = x$ and $y' = \sqrt{3}y$, the equation turns out to be $v_{x'x'} + v_{y'y'} - 44v = 0$. □

6. (a) It is hyperbolic since its discriminant $\mathscr{D} = (1/2)^2 > 0$;

   (b) Set $v = u_y$, we have $3v + v_x = 0$ which implies $v(x, y) = f(y)e^{-3x}$ and thus $u(x, y) = F(y)e^{-3x} + g(x)$, where F, g are arbitrary (differential) functions.

   (c) Setting $y = 0$ yields

$$e^{-3x} = u(x, 0) = F(0)e^{-3x} + g(x)$$
$$0 = u_y(x, 0) = F'(0)e^{-3x}.$$

   Therefore,

$$u(x, y) = (F(y) + 1 - F(0))e^{-3x},$$

   where $F(y)$ satisfy $F'(0) = 0$. By setting $F(y) = ny^2, n = 1, 2, \cdots$, we obtain infinitely many solutions $u(x, y) = (ny^2 + 1)e^{-3x}$ of the problem. □